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Note

Maximum induced matchings in graphs

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Abstract

We provide a formula for the number of edges of a maximum induced matching in a graph. As applications, we give some structural properties of $(k+1)K_2$-free graphs, construct all $2K_2$-free graphs, and count the number of labeled $2K_2$-free connected bipartite graphs.

1. Introduction

All graphs in this paper are understood to be finite, undirected, without loops, multiple edges or isolated vertices. In a graph G, $V = V(G)$ is the vertex set and $E = E(G)$ is the edge set of the graph G. For a vertex x of V, $N(x)$ is the set of neighbors of x. For a set $X \subseteq V$, $N(X)$ is the union of $N(x)$ for $x \in X$. The graph G' with vertex set V' and edge set E' is an *induced subgraph* of $G = (V,E)$ if $V' \subseteq V$ and $uv \in E'$, $u, v \in V'$ if and only if $uv \in E$. If the induced subgraph G' is a matching of k edges, then G' is an *induced* kK_2 or an *induced k-matching*. If G does not contain an induced $(k+1)K_2$, then G is a $(k+1)K_2$*-free* graph.

The first result about the $(k+1)K_2$-free graph can be found in [11] where a polynomial bound was given for the chromatic number of a $(k+1)K_2$-free graph in terms of the clique number. In contrast, there is no bound on the chromatic number of a graph in terms of its clique number in general. The extremal $(k+1)K_2$-free bipartite graphs with maximum degree d were considered in [6]; the largest such bipartite graph has kd^2 edges, and the number of edges of a connected $(k+1)K_2$-free extremal bipartite graph is no more than $kd^2 - d$; furthermore, all the extremal bipartite graphs were constructed.

In this paper, we provide max–min formulas for the maximum size of an induced matching in a bipartite graph and in a general graph. When the size is 0, we obtain characterizations of $(k+1)K_2$-free bipartite graphs and general $(k+1)K_2$-free

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graphs. The special case of $2K_2$-free graphs was studied in [4,5,7–9]. The problem of determining the maximum number of edges in a $2K_2$-free graph with a given maximum degree, proposed by Bermond et al. in [1] and also by Erdős and Nesetril was solved by Chung et al. in [4] by deriving and applying structural properties of $2K_2$-free graphs. Based on our characterization of $(k+1)K_2$-free bipartite graphs, we can construct all $2K_2$-free bipartite graphs and enumerate the number of labeled $2K_2$-free bipartite graphs.

Let $S_1, S_2, \ldots, S_m$ be a collection of subsets of a finite set. The set $\{x_1, x_2, \ldots, x_m\}$ is said to be a *system of distinct representatives* (SDR) for the collection if $x_i \in S_i$ for $i = 1, 2, \ldots, m$, and $x_i \neq x_j$ for $i \neq j$. Now we formulate a slightly different problem: Let $S_1, S_2, \ldots, S_m$ be a collection of subsets of a finite set. The set $\{x_1, x_2, \ldots, x_m\}$ is called a *strong system of distinct representatives* (SSDR) for the collection if $x_i \in S_i - \bigcup_{j \neq i} S_j$, for $i = 1, 2, \ldots, m$. From a given finite collection Φ of subsets of a finite set S, two problems can be formulated:

(1) *Find the maximum subcollection $\Phi_1 \subseteq \Phi$ such that Φ_1 has an SDR.*

(2) *Find the maximum subcollection $\Phi_1 \subseteq \Phi$ such that Φ_1 has an SSDR.*

While there is a polynomial algorithm to solve the first problem, the second problem is NP-complete. In fact, the second problem is equivalent to the following problem:

Is there an induced matching of size k in a bipartite graph G?

Cameron [3] proved that this problem is NP-complete.

2. Characterization of $(k+1)K_2$-free graphs

Let $\alpha''(G)$ be the number of edges of a maximum induced matching in a graph G.

Theorem 2.1. *For any graph G, let*

$$f(G) = \max_{G'} \min\{|X'|:\ X' \subseteq X \text{ and } Y \subseteq N(X')\},$$

where G' is an induced bipartite subgraph with partite sets X, Y and no isolated vertices. Then $\alpha''(G) = f(G)$.

Proof. Fix an induced bipartite subgraph G' with partite sets X, Y and no isolated vertices. Let $X_1 = \{x_1, \ldots, x_p\} \subseteq X$ be such that

$$|X_1| = \min\{|X'| : X' \subseteq X \text{ and } Y \subseteq N(X')\}.$$

Then $N(x_i) \cap Y \not\subseteq (\bigcup_{j \neq i} N(x_j)) \cap Y$, for $i = 1, 2, \ldots, p$, by the minimality of $|X_1|$. Choose $y_i \in Y \cap N(x_i) - (\bigcup_{j \neq i} N(x_j))$. Then $\{x_i y_i : i = 1, 2, \ldots, p\}$ is an induced matching of G' and is also an induced matching of G. Therefore, $\alpha''(G) \geqslant f(G)$.

On the other hand, let $\{x_1 y_1, \ldots, x_k y_k\}$ be a maximum induced matching of G, where $k = \alpha''(G)$. Let $X = \{x_1, \ldots, x_k\}$ and $Y = \{y_1, \ldots, y_k\}$. Then $X \cup Y$ induces kK_2. If $X' \not\subseteq X$ and $x_i \notin X'$, then $y_i \notin N(X')$ as $\{x_1 y_1, \ldots x_k y_k\}$ is an induced matching of G. Hence, if $Y \subseteq N(X')$, then $X' = X$. So for this special G', $\min\{|X'| : X' \subseteq X$ and $Y \subseteq N(X')\} = k$. This shows that $\alpha''(G) \leqslant f(G)$. The proof is completed. □

The following is a restatement of Theorem 2.1.

Corollary 2.2. *A graph G is $(k+1)K_2$-free if and only if for every induced bipartite subgraph having partite sets X and Y and no isolated vertices, there exists an $X' \subseteq X$ with $|X'| \leqslant k$ and $Y \subseteq N(X')$.*

If we restrict Theorem 2.1 and Corollary 2.2 to the class of bipartite graphs, we will obtain similar but simple results. The proof is also similar.

Theorem 2.3. *For a bipartite graph G having partite sets X and Y and no isolated vertices,*

$$\alpha''(G) = \max_{X''} \min\{|X'| : X' \subseteq X'' \subseteq X \text{ and } N(X') = N(X'')\}.$$

The proof is similar to the proof of Theorem 2.1. Like Theorem 2.1, this theorem has a useful rephrasing.

Corollary 2.4. *A bipartite graph G having partite sets X and Y and no isolated vertices is $(k+1)K_2$-free if and only if for any $X'' \subseteq X$, there exists $X' \subseteq X''$ such that $|X'| \leqslant k$ and $N(X') = N(X'')$.*

The proofs of the following corollaries are direct from Corollary 2.4. The case that $k = 1$ of Corollary 2.5 is also a corollary of Theorem 1 in [3]. And Corollary 2.6 is well-known and has been used in many contexts.

Corollary 2.5. *If G is a $(k+1)K_2$-free bipartite graph without isolated vertices, then each partite set of G contains a set of at most k vertices whose neighborhood contains the other partite set.*

Corollary 2.6. *A bipartite graph G without isolated vertices is $2K_2$-free if and only if for any two vertices u and v in the same color class, either $N(u) \subseteq N(v)$ or $N(v) \subseteq N(u)$.*

Corollary 2.7. *If $d(x) + d(y) < |V(G)|$ for every edge xy in a bipartite graph G without isolated vertices, then G contains an induced $2K_2$.*

3. A counting theorem

In this section, we consider labeled graph. We label a graph of order p with the integers from 1 through p. Two labeled graphs G_1 and G_2 are considered the *same* if they have the same labeled adjacency matrix.

Based on Corollary 2.6, we now construct all $2K_2$-free bipartite graphs with the fixed bipartition and without isolated vertices, and count the number of such labeled $2K_2$-free bipartite graphs.

Theorem 3.1. *Let $\boldsymbol{G}$ be the set of labeled $2K_2$-free bipartite graphs with given partite sets X and Y and no isolated vertices. If $|X| = m$ and $|Y| = n$, then*

$$|\boldsymbol{G}| = \sum_{t=1}^{\min(m,n)} t!^2 S(m,t)S(n,t),$$

where $S(r,t)$ (the Stirling number of the second kind, [9, p. 232]*) is the number of partitions of an r-set into t-parts.*

Proof. We note that $t!S(m,t)$ is the number of ordered partition of an m-set into t nonempty subsets. Hence it suffices to establish a bijection between $\boldsymbol{G}$ and the set of pairs of ordered partitions of X and Y into the same number of parts. If $G \in \boldsymbol{G}$, then by Corollary 2.6, the neighborhoods of the vertices in X form a chain $Y_t \subset Y_{t-1} \subset \cdots \subset Y_2 \subset Y_1 = Y$ by inclusion. Putting vertices with identical neighborhoods into the same block establishes an ordered partition of X into t parts. Because G has no isolated vertices, $Y_1 = Y$ and $Y_t \neq \emptyset$. Hence $Y_t, Y_{t-1} - Y_t, \ldots, Y_1 - Y_2$ is an ordered partition of Y into t parts. This process is reversible; given ordered partitions of X and Y with the same number of parts, we can retrieve the unique $G \in \boldsymbol{G}$ that generates it. □

Finally, we can count the labeled $2K_2$-free connected bipartite graphs.

Corollary 3.2. *The number of labeled $2K_2$-free bipartite graphs on n vertices is*

$$\frac{1}{2}\sum_{m=1}^{n-1}\binom{n}{m}\sum_{t=1}^{\min(m,n)} t!^2 S(m,t)S(n,t).$$

Proof. There are $\binom{n}{m}$ ways to distribute the labels to m vertices in one part and $n-m$ vertices in another part. Each resulting bipartite graph is counted twice. □

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